Corner Function Analysis of Microstrip Transmission Lines

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***Abstract*—A new method of analysis of microstrip transmission lines using corner functions (eigenfunctions) is presented. Assuming the TEM mode propagation for the shielded microstrip line structure, solutions are set up in a series of corner functions, which isolates the singularity at the edge of the strip. The boundary conditions are satisfied by using a method of successive integration of boundary errors. Numerical results are obtained for the charge distribution from which the microstrip line characteristic impedance is determined. Excellent agreement with published theoretical results and experimental data is obtained. The numerical results clearly bring out the power of the method.**

I. Introduction

THERE has been a considerable amount of interest since 1965 by several authors in the analysis of microstrip transmission lines. A large number of semianalytical and numerical methods have been applied for the solution of microstrip line problem. The principal techniques used are modified conformal mapping [1], finite-difference method [2], and the variational method [3], [4]. The analysis of microstrip line problem has posed difficulties due to the presence of singularities arising from the inhomogeneity in the structure and the discontinuity between the conductor and the dielectric surface (Fig. 1). The general treatment of microstrip transmission lines has been attempted by many workers [5]–[10] of which [5], [6], [10] deal with the singularities. Some of the recent investigations of harmonic boundary value problem in solid mechanics using "corner functions" provide an attractive method for analyzing problems with singularities [11], [12]. The "corner function" technique has recently been applied successfully for the solution of strip transmission line problem [13], [14]. In the present work, a similar approach is followed and an exact formulation to the solution of harmonic mixed boundary value problems in the presence of singularities is presented using the direct method of analysis employing corner functions. The solution is then specialized to tackle the microstrip transmission line problem.

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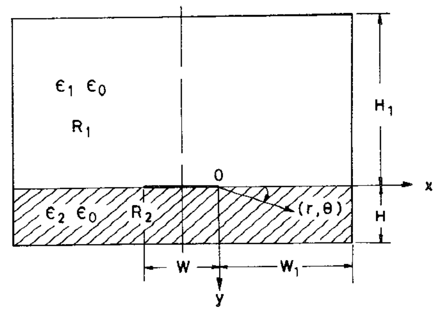

Fig. 1. Microstrip line in a shielded box.

II. Corner Function Analysis

Consider a two field problem in the form of an arbitrary sector with the regions $R1$ and $R2$ as shown in Fig. 2. The chosen coordinate system has its origin at the apex O. In each region, the boundary conditions on the edges OA and OC and the interface conditions on OB are all homogeneous.

In the present field problem, the two regions are governed by the Laplace's equation,

$$\nabla^2\Phi_i=0, \qquad i=1,2 \tag{1}$$

where i denotes the number of regions. Solutions are set up for each region separately in a series of 'corner functions' located at O, which identically satisfy the homogeneous boundary conditions along the edges OA and OC and the interface conditions on OB.

For a typical problem, boundary conditions on the edges and interface conditions will be

$$\Phi_1=0, \qquad \text{on } OA \tag{2}$$

$$\Phi_2=0, \qquad \text{on } OC \tag{3}$$

and

$$\left.\begin{aligned}\Phi_1 &= \Phi_2 \qquad (4)\\ \epsilon_1\frac{\partial\Phi_1}{\partial n} &= \epsilon_2\frac{\partial\Phi_2}{\partial n} \qquad (5)\end{aligned}\right\}, \qquad \text{on } OB$$

where ϵ_1 and ϵ_2 are the relative dielectric constants of the medium. Then the solutions can be written down in polar trigonometric form as

$$\Phi_1=r^{\lambda_m}F1(\lambda_m,\theta_1) \tag{6}$$

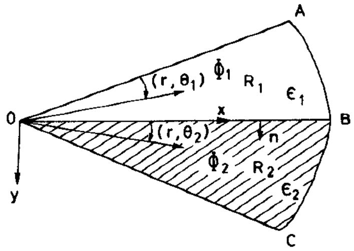

Fig. 2. Arbitrary sector having two dielectrics.

and

$$\Phi_2 = r^{\lambda_m} F2(\lambda_m, \theta_2) \tag{7}$$

respectively, for regions $R1$ and $R2$, where $F1$ and $F2$ are trigonometric functions. Because of the continuity across the interface, the sequence of eigenvalues λ_m is identical in both regions $R1$ and $R2$. The linear homogeneous equations arising from all the radial boundaries (edges and interfaces) yield a characteristic equation for the determination of λ_m. λ_m's in general are complex constants. In a field problem governed by the Laplace's equation, the potential Φ is always a positive quantity. Hence all $(\mathrm{Re}\lambda_m)$'s are positive. If any of the $(\mathrm{Re}\lambda_m)$'s is less than unity $(0 \leqslant \mathrm{Re}\lambda_m \leqslant 1)$, correspondingly a singularity in $\partial\Phi/\partial r$ of the order $r^{\mathrm{Re}\lambda_m - 1}$ arises. When an $(\mathrm{Re}\lambda_m)$ equals unity, the homogeneous boundary and interface conditions may yield a finite $\partial\Phi/\partial r$; however, in such cases, a logarithmic singularity may arise. If the lowest value of $(\mathrm{Re}\lambda_m)$ is above unity, no singularity is possible at the corner.

III. Analysis of Microstrip Lines

Fig. 1 shows the simple geometry of an inhomogeneous microstrip line shielded in a box. W is the width of the strip conductor on the dielectric substrate of thickness H, W_1 is the distance of the side wall from the edge of the strip, H_1 is the distance of the top ground plane from the strip and ϵ_1 and ϵ_2 are the dielectric constants of the two media. In the analysis of the above microstrip line problem, Laplace's equation (1) is considered as the governing equation for both the regions. Quasi-TEM propagation is assumed and the characteristic parameters are evaluated by considering the strip to be of infinitesmally small thickness.

As discussed above, the solutions for the Laplace's equation are set up in a series of corner functions as

$$\Phi_1 = \sum_m A_m \gamma^{\lambda_m} \cos\lambda_m\theta_1 + \sum_m B_m \gamma^{\lambda_m} \sin\lambda_m\theta_1 \tag{8}$$

and

$$\Phi_1 = \sum_m C_m \gamma^{\lambda_m} \cos\lambda_m\theta_2 + \sum_m D_m \gamma^{\lambda_m} \sin\lambda_m\theta_2 \tag{9}$$

where A_m, B_m, C_m, and D_m are arbitrary constants and λ_m's are the eigenvalues to be determined from the homogeneous part of the boundary conditions on the strip conductor OA (Fig. 3), viz.,

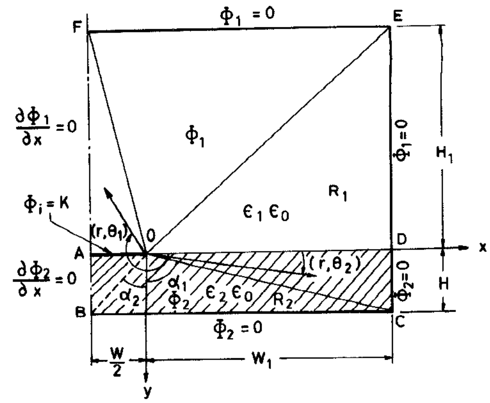

Fig. 3. Half-section of the microstrip line-scheme for the analysis.

$$\Phi_1|_{\theta_1=\pi} = \Phi_2|_{\theta_2=-\pi} = 0 \tag{10}$$

and the interface conditions OD between the two dielectrics

$$\Phi_1|_{\theta_1=2\pi} = \Phi_2|_{\theta_2=0} \tag{11}$$

and

$$\epsilon_1 \frac{\partial\Phi_1}{\partial y}\bigg|_{\theta_1=2\pi} = \epsilon_2 \frac{\partial\Phi_2}{\partial y}\bigg|_{\theta_2=0} \tag{12}$$

The potential of the strip conductor K which is excluded in (10) will be added to the functions after evaluation of the eigenvalues λ_m.

The eigenvalues λ_m's are now determined from the following characteristic equations obtained by applying (10)–(12) in (8) and (9) accordingly.

$$\cos\lambda_m\pi = 0, \qquad \lambda_m = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots \tag{13}$$

and

$$\sin\lambda_m\pi = 0, \qquad \lambda_m = 1, 2, 3, \cdots \tag{14}$$

and the satisfaction of the interface conditions leads to

$$C_m = -A_m \tag{15}$$

and

$$D_m = \frac{\epsilon_1}{\epsilon_2} B_m. \tag{16}$$

Substituting (13) to (16) into (8) and (9) and introducing the potential K of the strip conductor in the solution, the final solution is obtained as

$$\Phi_1 = K + \sum_{m=1,3,5,\cdots} A_m r^{m/2} \cos\frac{m\theta_1}{2} + \sum_{m=2,4,6,\cdots} B_m r^{m/2} \sin\frac{m\theta_1}{2} \tag{17}$$

and

$$\Phi_2 = K - \sum_{m=1,3,5,\cdots} A_m r^{m/2} \cos\frac{m\theta_2}{2} + \sum_{m=2,4,6\cdots} \frac{\epsilon_1}{\epsilon_2} B_m r^{m/2} \sin\frac{m\theta_2}{2}. \quad (18)$$

Equations (17) and (18) contain the arbitrary constants A_m and B_m which are to be determined by satisfying the boundary conditions in the respective regions. Due to symmetry about the center line BF, only one half of the whole region is to be considered. A_1, the first term in the series corresponds to the singular term of the solution. In this analysis A_1 will give the strength of the singularity on the edge of the strip.

IV. Boundary Conditions

By the consideration of the microstrip line symmetry (Fig. 3), boundary conditions for each region are given below.

Region R1

$$\frac{\partial\Phi_1}{\partial x} = 0 \text{ on } AF, \quad x = -\frac{W}{2}, \quad -H_1 \leqslant y \leqslant 0 \quad (19)$$

$$\Phi_1 = 0 \text{ on } EF, \quad y = -H_1, \quad -\frac{W}{2} \leqslant x \leqslant W_1 \quad (20)$$

$$\Phi_1 = 0 \text{ on } DE, \quad x = W_1, \quad -H_1 \leqslant y \leqslant 0. \quad (21)$$

Region R2

$$\Phi_2 = 0 \text{ on } CD, \quad x = W_1, \quad 0 \leqslant y \leqslant H \quad (22)$$

$$\Phi_2 = 0 \text{ on } BC, \quad y = H_1, \quad -\frac{W}{2} \leqslant x \leqslant W_1 \quad (23)$$

$$\frac{\partial\Phi_2}{\partial x} = 0 \text{ on } AB, \quad x = -\frac{W}{2}, \quad 0 \leqslant y \leqslant H. \quad (24)$$

V. Numerical Evaluation of the Constants

The boundary conditions can be satisfied by any one of the well-known techniques available in literature [15]. Here a new method of satisfying the boundary conditions, viz., a method of successive integration of boundary errors has been utilized for the evaluation of the unknown constants. In this method, the error R on each of the boundary is successively integrated and equated to zero, viz.,

$$\int_{s_1}^{s_2}\int_{s_1}^{s}\cdots\int_{s_1}^{s} R\cdot ds\cdot ds\cdots ds = 0 \quad (25)$$

where s_1 and s_2 are the limits of integration and s is the coordinate along the edge. Each integration applied on each boundary will be able to provide one equation in the system of simultaneous equations in terms of the arbitrary constants A_m's and B_m's. By increasing the number of integration in a systematic manner, the arbitrary constants are determined for different orders of approximation (involving different number of equations) and a convergence study is made. It can be shown that (25) can be reduced to a single integral by the following identity:

$$\int_{s_1}^{s_2}\int_{s_1}^{s}\cdots\int_{s_1}^{s}[R=f(s)](ds)^{k+1} = \frac{1}{k!}\int_{s_1}^{s_2}(s_2-s)^k[R=f(s)]\,ds \quad (26)$$

where $k = 0, 1, 2, \cdots$ denotes the number of integrations. Thus it is seen that the method is equivalent to an orthogonality method in which the error function is made orthogonal to a simple polynomial series.

It may be noted that as distinct from a least square error method of satisfying the boundary conditions, the present successive integration method ensures self-equilibration of boundary errors. Due to this advantage rapid convergence of the solution is achieved.

The above method is used for the satisfaction of the boundary conditions on all the boundaries in both the regions. The method is now illustrated for the boundary BC $(y=H)$ in the region $(\pi/2-\alpha_1) \leqslant \theta_2 \leqslant (\pi/2+\alpha_2)$, where $\alpha_1 = \tan^{-1} W_1/H$ and $\alpha_2 = \tan^{-1}(W/2)/H$ as depicted in Fig. 3. With this information, the boundary error equation can be obtained from (18)

$$\Phi_2|_{y=H} = K - \sum_{m=1,3,5,\cdots} A_m (H\operatorname{cosec}\theta_2)^{m/2}\cos\frac{m\theta_2}{2} + \sum_{m=2,4,6,\cdots}\frac{\epsilon_1}{\epsilon_2} B_m (H\operatorname{cosec}\theta_2)^{m/2}\sin\frac{m\theta_2}{2} = 0. \quad (27)$$

Applying the identity (26) the error equation becomes

$$\frac{1}{k!}\int_{(\pi/2-\alpha_2)}^{(\pi/2+\alpha_2)}\left(\frac{\pi}{2}+\alpha_2-\theta_2\right)^k \cdot\left[K - \sum_{m=1,3,5,\cdots} A_m (H\operatorname{cosec}\theta_2)^{m/2}\cos\frac{m\theta_2}{2} + \sum_{m=2,4,6,\cdots}\frac{\epsilon_1}{\epsilon_2} B_m (H\operatorname{cosec}\theta_2)^{m/2}\sin\frac{m\theta_2}{2}\right] d\theta_2 = 0. \quad (28)$$

Above (28) is numerically integrated by the use of Gauss–Laguerre quadrature polynomials which reduces the limit from $-\frac{1}{2}$ to $+\frac{1}{2}$ and form one of the sets of the simultaneous equations. Similar sets of equations are generated by the above method for each of the boundaries. The simultaneous equations are then solved by a suitable method and the arbitrary constants determined.

VI. Determination of the Microstrip Line Impedance

The characteristic impedance of the microstrip line is evaluated in the following way using the constants determined from the corner function series. The charge-distribution on the strip has been calculated at a number of points all along the strip by computing the line integrals of the electric field normal to the boundary. The characteristic impedance has been determined from the knowledge of the charge-distribution. In order to evaluate the microstrip line characteristic impedance Z, for TEM mode approximation, free space capacitance C_a is first evaluated for the homogeneous air medium and then

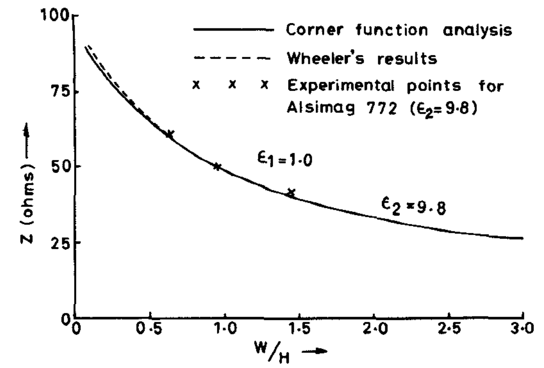

Fig. 4. Characteristic impedance of the microstrip line.

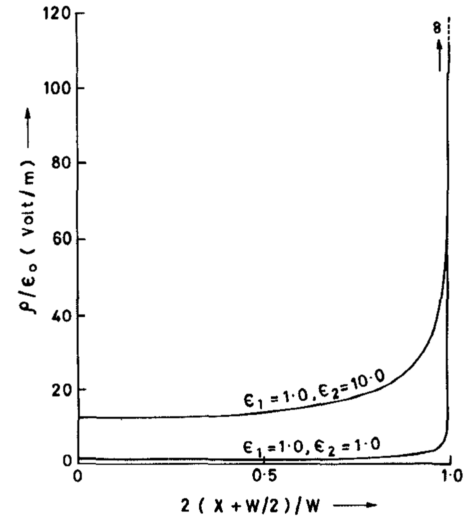

Fig. 5. Charge distribution on the strip conductor ($W/H=1.0$, $W_1=3.0$, $H_1=3.5$).

capacitance C_d in the presence of inhomogeneously filled dielectric as

$$Z=\frac{1}{c\sqrt{C_d C_a}} \tag{29}$$

where $c=2.997925\times10^8$ m/s, velocity of light in free space.

VII. Results and Discussions

Numerical results are obtained for microstrip line structure with dielectric constants of homogeneous air medium and $\epsilon_1=1.0$ and $\epsilon_2=9.8$ for various values of W/H ratio. The boundaries BC and EF are suitably divided and each division treated separately for purposes of integration so that enough weightage is given to regions close to the strip conductor. Results are obtained for the three orders of approximation with $k=0$, 1, and 2 involving matrix orders of 10, 20, and 30. Convergence studies show that the convergence is rapid for all the cases and the trend is monotonic for small values of W/H and is slightly oscillatory for large values of W/H. In all cases, comparison with earlier results [1] show that the 20 term values (in many cases even 10 term values) are highly accurate, the difference between the two being less than 1 percent. The errors on the boundary are also studied and found to be less than 0.001 percent in the ground plane regions close to the strip.

The characteristic impedance of the microstrip line is presented in Fig. 4 as a function of W/H ratio for the dielectric constant 9.8 of the substrate. Wheeler's results by conformal mappling method is also shown for comparison purpose. Fig. 4 shows excellent agreement between the present result with those of Wheeler's.

Experimental results of the characteristic impedance have been obtained for various values of W/H ratios on Alsimag 772 (dielectric constant = 9.8) substrates. Measurements have been carried out using TDR (Tektronix 7S12) and the results are shown in Fig. 4 which shows close agreement.

The charge-distribution on the strip at $y=0$ for the homogeneous air medium with $W/H=1.0$, $W_1=3.0$, and $H_1=3.5$ is estimated from the theoretical analysis and shown in Fig. 5. As expected, the charge-distribution indicates singularity at the edge of the strip. Fig. 5 also shows the charge-distribution of the microstrip line with $\epsilon_1=1.0$ and $\epsilon_2=10.0$, the other parameters remaining the same as above. This plot clearly shows the increase in total charge of the microstrip line structure with the insertion of the dielectric layer, however, the nature of singularity and the edge conditions remain unchanged.

Main features of the above analysis are summarized as follows:

1) By the method adopted here, it is possible to identify the nature of the singularity and isolate the singularity.

2) By the use of the successive integration method of satisfying the boundary conditions, apart from making the error on the boundary to a minimum the error is also made to be self-equilibrating. Rapid convergence is obtained with only a few terms of the series (first order solution involving a 10×10 matrix). Convergence is monotonic for small values of W/H while it becomes slightly oscillatory for large W/H values.

3) By considering large values of H_1 and/or W_1, cases of no top wall and/or no side wall can be analyzed without difficulty.

4) No need for the inversion process as involved in the earlier studies [6].

5) Accurate results are obtained with little effort with the inclusion of the singularities.

6) The computational time taken by this approach (of the order of <15 s in DEC-system 10) is very little.

VIII. Conclusion

A rigorous method for the analysis of microstrip transmission lines using corner function has been presented. Successive integration method has been applied for the satisfaction of the boundary conditions which gives accurate results with only a few terms of the series. Results are presented for the microstrip line characteristic impedance for various values of W/H ratio, with $\epsilon_1 = 1.0$ and $\epsilon_2 = 9.8$ and compared with the available theoretical results and the measured experimental values, which shows very close agreement. The same method can be applied to solve many other problems like suspended microstrip line and slot line.

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